Lecture 7

Einstein equations

- Poll
  - Book ✓
  - Long exercises; essential?
  - "Trivial" ×
  - pass slides too quickly

Recap

Where were we?

Covariant derivative of a vector

$$\boxed{\underbrace{\nabla_\mu V^\nu}_{\text{(1,1) tensor}} = \underbrace{\partial_\mu V^\nu}_{\text{not a tensor}} + \overset{\text{not a tensor}}{\underset{\text{Connection coefficients}}{\Gamma^\nu_{\mu\lambda}}} V^\lambda}$$

The Levi-Civita connection

There is a unique torsion free and metric compatible connection.

$$T^\lambda_{\mu\nu} = 2\,\Gamma^\lambda_{[\mu\nu]} = 0 \qquad\qquad \nabla_\rho g_{\mu\nu} = 0$$

$$\Updownarrow$$

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)}$$

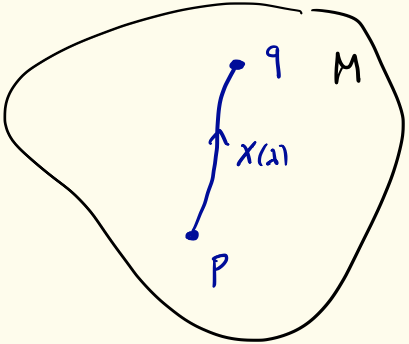
$$\Rightarrow \boxed{\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)}$$

Christoffel symbols

Geodesics

A timelike geodesic maximizes the proper time between 2 points:

$$\tau = \int d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$



Christoffel symbols

$\bullet \equiv \frac{d}{d\tau}$ ← proper time

$$\delta\tau = 0 \Rightarrow \boxed{\ddot{x}^\sigma + \Gamma^\sigma_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0} \Leftrightarrow \frac{D}{d\tau}\dot{x}^\sigma = \dot{x}^\mu\nabla_\mu\dot{x}^\sigma = 0$$

geodesic equation

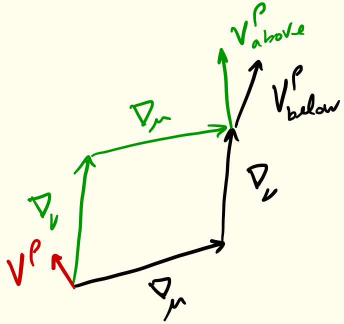
The tangent vector is parallel transported.

The Riemann Curvature Tensor

In the presence of curvature covariant derivatives do not commute:

$$\boxed{[\nabla_\mu, \nabla_\nu] V^\rho = \underset{\text{Riemann tensor}}{R^\rho{}_{\sigma\mu\nu}} V^\sigma}$$

(assuming torsion $= 0$)



$$\boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}}$$

Properties of the Riemann tensor

$$\boxed{R_{\rho\sigma\mu\nu} = - R_{\rho\sigma\nu\mu}}$$

$$\boxed{R_{\rho\sigma\mu\nu} = - R_{\sigma\rho\mu\nu}}$$

$$\boxed{R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}}$$

$$\boxed{R_{\rho[\sigma\mu\nu]} = 0}$$

$$\boxed{\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0}$$

Bianchi identity

Ricci tensor :

$$\boxed{R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}} \qquad R_{\mu\nu} = R_{\nu\mu}$$

Ricci scalar :

$$\boxed{R \equiv g^{\mu\nu} R_{\mu\nu}}$$

Weyl tensor :

$$\boxed{C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}\right) + \frac{2}{(n-2)(n-1)} g_{\rho[\mu} g_{\nu]\sigma} R}$$

↳ Same symmetries of Riemann tensor with all traces removed (check!)

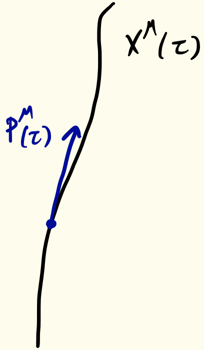
Einstein tensor :

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}} \qquad \boxed{\nabla^{\mu} G_{\mu\nu} = 0}$$

Symmetries and Killing vectors

Geodesic equation : $p^\mu \nabla_\mu p^\nu = 0$

$$\left[\begin{array}{ll} p^\mu = m\dfrac{dx^\mu}{d\tau} & \text{massive} \\ p^\mu = \dfrac{dx^\mu}{d\lambda} & \text{massless} \end{array}\right]$$



Killing vector field

$$\boxed{\nabla_{(\mu} K_{\nu)} = 0} \quad \Rightarrow \quad K_\mu p^\mu \text{ is conserved along the geodesic}$$

Killing's equation

$$\frac{d}{d\tau}\left(K_\mu p^\mu\right) = 0$$

Symmetries and Killing vectors

$$J^\mu \equiv k_\nu \underset{\nwarrow \text{ energy-mom. tensor}}{T^{\mu\nu}}$$

$$\boxed{\nabla_{(\mu} \overset{\text{Killing vector field}}{\overset{\downarrow}{K_{\nu)}}} = 0}$$

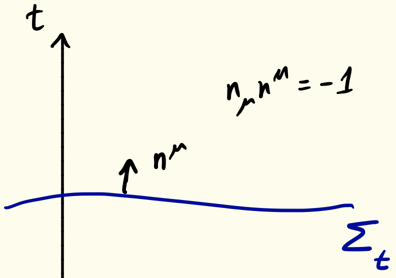
$$\nabla_\mu J^\mu = \underbrace{\left(\nabla_{(\mu} k_{\nu)}\right)}_{\overset{\shortparallel}{0}\ \text{Killing's eq.}} T^{\mu\nu} + k_\nu \underbrace{\nabla_\mu T^{\mu\nu}}_{\overset{\shortparallel}{0}\ \text{cons. of } T^{\mu\nu}} = 0$$

$$\left[\begin{aligned} &\underbrace{X_{\mu\nu}}_{X_{(\mu\nu)} + X_{[\mu\nu]}} T^{\mu\nu} = \\ &\underset{\overset{\uparrow}{T^{\mu\nu} = T^{\nu\mu}}}{=} X_{(\mu\nu)} T^{\mu\nu} \end{aligned} \right]$$

$\Rightarrow$ There is a conserved quantity:

$$Q(t) = -\int_{\Sigma_t} d^3x \sqrt{\gamma}\, n_\mu J^\mu = -\int_{\Sigma_t} *\overset{\overset{\text{1-form}}{J_\mu dx^\mu}}{\overset{\shortparallel}{J}}$$

$$\frac{d}{dt} Q(t) = 0$$

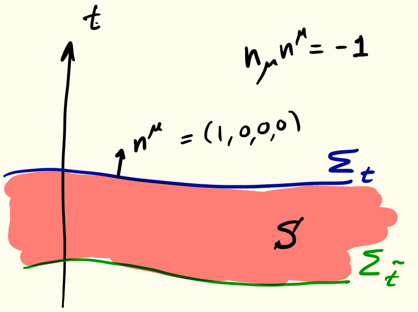


$n_\mu n^\mu = -1$

$$ds^2 = -dt^2 + \underbrace{\gamma_{ij}\, dx^i dx^j}_{ds^2_\Sigma}$$

Proof:

$$Q(t) = -\int_{\Sigma_t} d^3x \sqrt{\gamma}\, n_\mu J^\mu = -\int_{\Sigma_t} * \overset{\overset{\text{1-form}}{J_\mu dx^\mu}}{\overset{\shortparallel}{J}}$$



$$n_\mu n^\mu = -1$$

$$Q(t) - Q(\tilde t) = -\int_{\Sigma_t} *J + \int_{\Sigma_{\tilde t}} *J = -\int_{\partial S} *J \underset{\substack{\uparrow \\ \text{Stokes} \\ \text{theorem}}}{=} -\int_S d(*J) \underset{\uparrow}{=} 0$$

$$d(*J) = 0$$
$$\Leftrightarrow \nabla_\mu J^\mu = 0$$

$$\partial S = \Sigma_t \cup \Sigma_{\tilde t}$$

$$ds^2 = -dt^2 + \underbrace{\gamma_{ij}\, dx^i dx^j}_{ds^2_\Sigma}$$

$$n_\mu dx^\mu = -dt$$

$$Q(t) - Q(\tilde t) = \int_{\Sigma_t} d^3x \overset{\sqrt{-g}}{\overset{\shortparallel}{\sqrt{\gamma}}} J^0 - \int_{\Sigma_{\tilde t}} d^3x \sqrt{\gamma}\, J^0 = \int d^3x \int_{\tilde t}^{t} dx^0 \Big[\underbrace{\partial_0\left(\sqrt{-g}\, J^0\right) + \partial_i\left(\sqrt{-g}\, J^i\right)}_{\partial_\mu\left(\sqrt{-g} J^\mu\right) = \sqrt{-g}\, \nabla_\mu J^\mu} \Big] = 0$$

Examples :

$$ds^2 = -dt^2 + dx^i dx^j \delta_{ij}$$

→ $k = \frac{\partial}{\partial t} \quad \Rightarrow \quad Q = \int d^3x \, T^0{}_0 = -E$

→ $k = \frac{\partial}{\partial x^1} \quad \Rightarrow \quad Q = \int d^3x \, T^0{}_1 = P_1$

→ $k = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} \quad \Rightarrow \quad Q = L_{12} = \int d^3x \left(T^0{}_2 \, x^1 - T^0{}_1 \, x^2 \right)$

Plan for today

- Maximally symmetric spaces
- Einstein Equivalence Principle
- Einstein Equations
- Einstein - Hilbert Action

Maximally Symmetric Spaces are spaces with the maximal $\underbrace{\text{number of isometries}}_{\substack{\| \\ \#\text{ of lin. ind.} \\ \text{Killing vector fields}}}$ for a given dimension.

Example: $\mathbb{R}^n$

$$\#\text{ of isometries} = \underset{\substack{\uparrow \\ \text{transl.}}}{n} + \underset{\substack{\uparrow \\ \text{rot.}}}{\frac{n(n-1)}{2}} = \frac{n(n+1)}{2}$$

$$\Rightarrow \boxed{R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}\left(g_{\rho\mu}\, g_{\sigma\nu} - g_{\rho\nu}\, g_{\sigma\mu}\right)} \;,\; R = \text{const.} \begin{cases} = 0 & \mathbb{R}^n \\ > 0 & \text{sphere} \\ < 0 & \text{hyperboloid} \end{cases}$$

Maximally Symmetric Spaces are spaces with the maximal $\underbrace{\text{number of isometries}}_{\substack{\| \\ \text{\# of lin. ind.} \\ \text{Killing vector fields}}}$ for a given dimension.

Example : $\mathbb{R}^n$

$$\# \text{ of isometries} = \underset{\substack{\uparrow \\ \text{translations}}}{n} + \underset{\substack{\uparrow \\ \text{rotations}}}{\frac{n(n-1)}{2}} = \frac{n(n+1)}{2}$$

$$\Rightarrow \boxed{R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right)} \quad , \quad R = \text{const.} \begin{cases} = 0 & \mathbb{R}^n \\ > 0 & \text{spheres} \\ < 0 & \text{hyperboloids} \end{cases}$$

Proof : In normal coordinates at p, $g_{\hat\mu\hat\nu} = \eta_{\hat\mu\hat\nu}$ and the Riemann tensor must be invariant under rotations around p. Thus, it must be constructed from $\eta_{\hat\mu\hat\nu}$, $\epsilon_{\hat\mu_1 \dots \hat\mu_n}$, $\delta^{\hat\nu}_{\hat\mu}$. $\Rightarrow$ $R_{\rho\sigma\mu\nu} \propto g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}$

Equivalence Principle

Experimentally, we observe that:

$$\underset{\substack{\text{inertial}\\ \text{mass}}}{m_i} = \underset{\text{grav. mass}}{m_g}$$

$\vec{F} = m_i \vec{a}$

$\vec{F}_g = -m_g \vec{\nabla} \Phi$ ↙ grav. pot.

$\Rightarrow \exists$ inertial or "free-falling" trajectories for $\underbrace{\text{unaccelerated particles}}_{\text{subject only to gravity}}$

Equivalence Principle

Experimentally, we observe that: $$\underset{\substack{\text{inertial}\\\text{mass}\\ \vec{F} = m_i \vec{a}}}{m_i} = \underset{\substack{\text{gravitational}\\\text{mass}\\ \vec{F} = -m_g \vec{\nabla} \overset{\text{grav. potential}}{\Phi}}}{m_g}$$

$\Rightarrow$ $\exists$ inertial or "free-falling" trajectories for $\underbrace{\text{unaccelerated particles}}_{\text{subject only to gravity}}$

This suggests that gravity is not a "force".
Instead, it describes how different inertial trajectories move with respect to each other.

Gravity = Geometry ?

Einstein Equivalence Principle (EEP)

"In small enough regions of spacetime, the laws of Physics reduce to those of Special Relativity; it is impossible to detect the existence of a gravitational field by means of local experiments."

↓

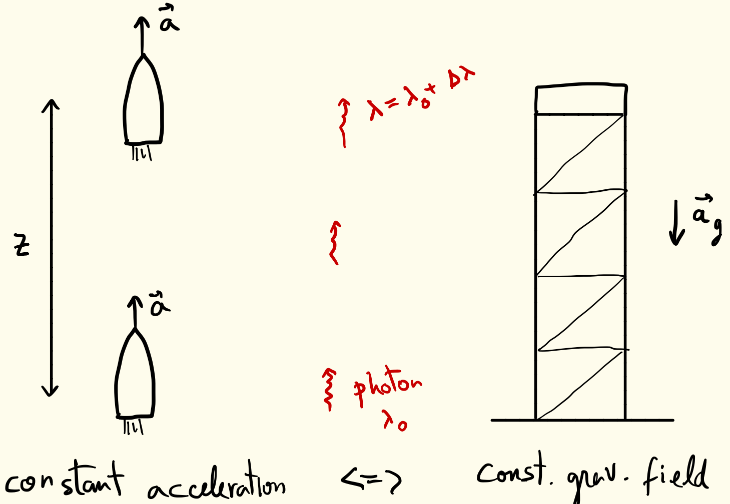
In a finite region, one can observe tidal effects if the grav. field is not homogeneous.

Einstein Equivalence Principle (EEP)

"In small enough regions of spacetime, the laws of Physics reduce to those of Special Relativity; it is impossible to detect the existence of a gravitational field by means of local experiments."

$\downarrow$ In a finite region, one can observe tidal effects if the grav. field is not homogeneous.

Example:



$$\boxed{\frac{\Delta\lambda}{\lambda_0} = \overset{\frac{\Delta v}{c}}{\overset{\parallel}{\frac{az}{c^2}}} = \frac{a_g z}{c^2}}$$

Doppler effect

Grav. redshift

$\Delta t = \frac{z}{c}$ $\quad (v \ll c)$

$\Delta v = a \Delta t$

Newtonian Gravity

grav. pot.

① $\boxed{\vec{a} = -\vec{\nabla}\Phi}$

mass density

② $\boxed{\nabla^2\Phi = 4\pi G \rho}$

Newton's constant

How to generalize these equations?

EEP →

$$\begin{tikzcd} \text{gravity} \arrow[d] & = & \text{geometry} \arrow[d] \\ \text{SR} & = & \text{normal coord.} \end{tikzcd}$$

locally

⇒ Minimal-coupling principle

||

just write SR laws in coord. inv. form!

Example: particles move in straight lines

$$\frac{d^2x^\mu}{d\tau^2} = 0 \iff \frac{dx^\nu}{d\tau}\,\partial_\nu\,\frac{dx^\mu}{d\tau} = 0 \longrightarrow \boxed{\frac{dx^\nu}{d\tau}\,\nabla_\nu\,\frac{dx^\mu}{d\tau} = 0}$$

geodesic equation.

Lecture 8

Einstein equations (part 2)

Recap

Where were we?

Covariant derivative of a vector

$$\boxed{\underbrace{\nabla_\mu V^\nu}_{(1,1)\ \text{tensor}} = \underbrace{\partial_\mu V^\nu}_{\substack{\text{not}\\ \text{a tensor}}} + \underset{\substack{\text{Connection}\\ \text{coefficients}}}{\overset{\text{not a tensor}}{\Gamma^\nu_{\mu\lambda}}} V^\lambda}$$

The Levi-Civita connection

There is a unique torsion free and metric compatible connection.

$$T^\lambda_{\mu\nu} = 2\,\Gamma^\lambda_{[\mu\nu]} = 0 \qquad\qquad \nabla_\rho g_{\mu\nu} = 0$$

$$\Updownarrow$$

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)}$$

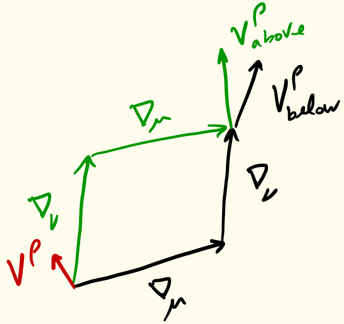
$$\Rightarrow \quad \boxed{\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)} \quad \text{Christoffel symbols}$$

The Riemann Curvature Tensor

In the presence of curvature covariant derivatives do not commute:

$$\boxed{[\nabla_\mu, \nabla_\nu] V^\rho = \underset{\text{Riemann tensor}}{R^\rho{}_{\sigma\mu\nu}} V^\sigma}$$

(assuming torsion = 0)



$$\boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}}$$

Ricci tensor :

$$\boxed{R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}} \qquad R_{\mu\nu} = R_{\nu\mu}$$

Ricci scalar :

$$\boxed{R \equiv g^{\mu\nu} R_{\mu\nu}}$$

Weyl tensor :

$$\boxed{C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}\right) + \frac{2}{(n-2)(n-1)} g_{\rho[\mu} g_{\nu]\sigma} R}$$

Same symmetries of Riemann tensor with all traces removed (check!)

Einstein tensor :

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}} \qquad \boxed{\nabla^{\mu} G_{\mu\nu} = 0}$$

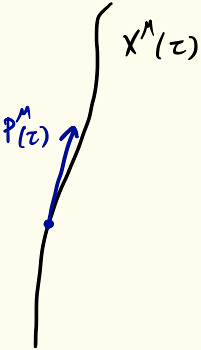
Symmetries and Killing vectors

Killing vector field

$$\boxed{\nabla_{(\mu} K_{\nu)} = 0}$$

Killing's equation

$\Rightarrow$ $k_\mu p^\mu$ is conserved along the geodesic



$$J^\mu \equiv K_\nu T^{\mu\nu}$$

energy-mom. tensor

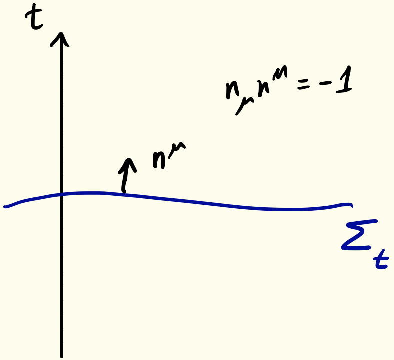
← conserved current $\nabla_\mu J^\mu = 0$

$\Rightarrow$ There is a conserved quantity:

1-form

$J_\mu dx^\mu$

$$Q = -\int_{\Sigma_t} d^3x \sqrt{\gamma}\, n_\mu J^\mu = -\int_{\Sigma_t} * J$$



$\underbrace{\text{Maximally Symmetric Spaces}}$ are spaces with the maximal $\underbrace{\text{number of isometries}}_{\substack{\shortparallel \\ \#\text{ of lin. ind.} \\ \text{Killing Vector fields}}}$ for a given dimension.

Example : $\mathbb{R}^n$

$$\# \text{ of isometries} = \underset{\substack{\uparrow \\ \text{translations}}}{n} + \underset{\substack{\uparrow \\ \text{rotations}}}{\frac{n(n-1)}{2}} = \frac{n(n+1)}{2}$$

$$\Rightarrow \boxed{R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right)} \quad , \quad R = \text{const.} \begin{cases} = 0 & \mathbb{R}^n \\ > 0 & \text{spheres} \\ < 0 & \text{hyperboloids} \end{cases}$$

Equivalence Principle

Experimentally, we observe that: $\underset{\text{inertial mass}}{m_i} = \underset{\text{gravitational mass}}{m_g}$

$\vec{F} = m_i \vec{a}$ $\qquad$ $\vec{F} = -m_g \vec{\nabla} \Phi$

grav. potential

$\Rightarrow$ $\exists$ inertial or "free-falling" trajectories for $\underbrace{\text{unaccelerated particles}}_{\text{subject only to gravity}}$

Einstein Equivalence Principle (EEP)

"In small enough regions of spacetime, the laws of Physics reduce to those of Special Relativity; it is impossible to detect the existence of a gravitational field by means of local experiments."

↓

In a finite region, one can observe tidal effects if the grav. field is not homogeneous.

Plan for today

- Scalar gravity
- Einstein Equations
- Einstein - Hilbert Action
- The Cosmological Constant

Newtonian Gravity

grav. pot.

$$① \quad \boxed{\vec{a} = -\vec{\nabla}\Phi}$$

mass density

$$② \quad \boxed{\nabla^2 \Phi = 4\pi G \rho}$$

Newton's constant

How to generalize these equations?

Scalar Gravity

$$\frac{dU^\mu}{d\tau} = -\partial^\mu \Phi$$

$$\boxed{\eta^{\mu\nu}\partial_\mu \partial_\nu \Phi = -4\pi G \, T_\mu{}^\mu}$$

$$\left(\text{or } T_{\mu\nu} U^\mu U^\nu\right)$$

Problem: $$U_\mu \frac{dU^\mu}{d\tau} = \frac{1}{2}\frac{d}{d\tau}(\overbrace{U_\mu U^\mu}^{-1}) = 0 = -U^\mu \partial_\mu \Phi = -\frac{d\Phi(x(\tau))}{d\tau}$$

Solution:

$$\boxed{\frac{dU^\mu}{d\tau} = -\underbrace{\left(\eta^{\mu\nu} + U^\mu U^\nu\right)}_{\text{Projector to } \perp \text{ to } U^\mu} \partial_\nu \Phi}$$

Scalar Gravity

$$\circledast \quad \boxed{\frac{dU^\mu}{d\tau} = -\left(\eta^{\mu\nu} + U^\mu U^\nu\right)\partial_\nu \Phi} \qquad \boxed{\eta^{\mu\nu}\partial_\mu \partial_\nu \Phi = -4\pi G\, T_\mu{}^\mu}$$

Exercise: Show that $\circledast$ follows from the actions

1) $$S_1 = -m \int d\lambda\, e^{\Phi(x(\lambda))} \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \quad , \quad \bullet \equiv \frac{d}{d\lambda}$$

2) $$S_2 = \frac{1}{2}\int d\lambda \left[\frac{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{E(\lambda)} - m^2 e^{2\Phi} E(\lambda)\right]$$

$$\frac{\delta S_2}{\delta E} = 0 \Rightarrow -\frac{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{E^2} - m^2 e^{2\Phi} = 0 \Leftrightarrow E = \frac{\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}{m} e^{-\Phi}$$

Replace E into S_2 to get S_1.

Scalar Gravity

$$\circledast\ \cancel{m_{in}}\ \boxed{\frac{dU^\mu}{d\tau} = -\left(\eta^{\mu\nu} + U^\mu U^\nu\right)\partial_\nu \Phi}\ \frac{m_{grav}}{m_{in}} \qquad \boxed{\eta^{\mu\nu}\partial_\mu \partial_\nu \Phi = -4\pi G\, T_\mu{}^\mu}$$

$$S = \frac{1}{2}\int d\lambda \left[\frac{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{E} - m^2 e^{2\Phi} E\right] , \qquad \delta S = 0 \implies \circledast$$

Problem 1: There is no light deflection in scalar gravity.

$$S = \frac{1}{2}\int d\lambda \left[\frac{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{E} - \underset{m=0}{\cancel{m^2 e^{2\Phi}}} E\right] \implies$$

photon trajectories are independent of Φ

Problem 2: $m_{inertial} = m_{grav}$ is not "natural"

$$S = \frac{1}{2}\int d\lambda \left[\frac{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{E} - m^2 e^{\alpha 2\Phi} E\right] \qquad \alpha \sim \frac{m_{grav}}{m_{inertial}}$$

Could be different for each particle

Newtonian Gravity

grav. pot.

① $\boxed{\vec{a} = -\vec{\nabla}\Phi}$

mass density

② $\boxed{\nabla^2\Phi = 4\pi G \rho}$

Newton's constant

How to generalize these equations?

EEP → gravity = geometry

↓ locally ↓

SR = normal coord.

⟹ Minimal-coupling principle

‖

just write SR laws in coord. inv. form!

Example: particles move in straight lines

$$\frac{d^2x^\mu}{d\tau^2} = 0 \iff \underbrace{\frac{dx^\nu}{d\tau}\partial_\nu\frac{dx^\mu}{d\tau}}_{\text{Not coord. invariant}} = 0 \rightarrow \boxed{\frac{dx^\nu}{d\tau}\nabla_\nu\frac{dx^\mu}{d\tau} = 0}$$

geodesic equation.

Newtonian limit

- Non - relativistic : $|\vec{v}| \ll c$
- Weak grav. field : $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$
- static : $\frac{\partial}{\partial t} h_{\mu\nu} = 0$

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \longrightarrow \frac{d^2 x^\mu}{d\tau^2} + \underbrace{\Gamma^\mu_{00}}_{\frac{1}{2} g^{\mu\nu}\left(\cancel{g_{\nu 0,0}} + \cancel{g_{0\nu,0}} - g_{00,\nu}\right)} \left(\frac{dx^0}{d\tau}\right)^2 = 0$$

static

$$= -\frac{1}{2} \eta^{\mu\nu} \partial_\nu h_{00} + O(h^2)$$

$\mu = 0$: $\frac{d^2 t}{d\tau^2} = 0$

$\mu = i$: $\frac{d^2 x^i}{d\tau^2} = \frac{1}{2} \left(\frac{dt}{d\tau}\right)^2 \partial_i h_{00}$

①

$$\Rightarrow \frac{d^2 x^i}{dt^2} = \frac{1}{2} \partial_i h_{00} = - \partial_i \Phi$$

$$\Rightarrow \boxed{g_{00} = -1 - 2\Phi}$$

Einstein Equation

How to generalize $\underset{\underset{g_{00}}{\Uparrow}}{\nabla^2 \Phi} = 4\pi G \underset{\underset{T_{00}}{\Uparrow}}{\rho}$?

Maybe $[\nabla^2 g]_{\mu\nu} \propto T_{\mu\nu}$? No, because $\nabla_\alpha g_{\mu\nu} = 0$.

Or $R_{\mu\nu} \propto T_{\mu\nu}$? No, because $\nabla^\mu T_{\mu\nu} = 0 \Rightarrow \underset{\text{not natural}}{\nabla^\mu R_{\mu\nu} = 0}$

Then, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \propto T_{\mu\nu}$? Yes !

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}} \overset{n=4}{\Longrightarrow} R - 2R = 8\pi G T_\mu{}^\mu$$

$$\Big\Updownarrow \; n=4$$

$$\boxed{R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T^\alpha{}_\alpha g_{\mu\nu} \right)}$$

Newtonian limit

- Non-relativistic : $|\vec{v}| \ll c$
- Weak grav. field : $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, $h_{00} = -2\Phi$
- static : $\frac{\partial}{\partial t} h_{\mu\nu} = 0$

$$R_{00} = R^{\mu}{}_{0\mu0} = R^{i}{}_{0i0}$$

$$R^{i}{}_{0j0} = \partial_j \Gamma^{i}_{00} - \underbrace{\partial_0 \Gamma^{i}_{j0}}_{\overset{\text{static}}{=} 0} + O(\Gamma^2) \quad , \quad \Gamma \propto \partial h$$

$$R_{00} \approx \partial_i \Gamma^{i}_{00} = \frac{1}{2} \partial_i \, g^{i\lambda} \left(\cancel{\partial_0 g_{0\lambda}} + \cancel{\partial_0 g_{0\lambda}} - \partial_\lambda g_{00} \right)$$

$$\approx -\frac{1}{2} \eta^{i\lambda} \partial_i \partial_\lambda h_{00} = -\frac{1}{2} \nabla^2 h_{00} = \nabla^2 \Phi$$

Newtonian limit

- Non-relativistic : $|\vec{v}| << c$, $p << \rho$
- Weak grav. field : $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| << 1$, $g_{00} \simeq -1-2\Phi$
- static : $\frac{\partial}{\partial t} h_{\mu\nu} = 0$

$$\Rightarrow \quad R_{00} \approx \nabla^2 \Phi$$

$$\Rightarrow \quad T_{\mu\nu} \simeq \rho\, U_\mu U_\nu \quad , \quad \text{rest frame} \quad U^\mu = (\overset{g_{00}U^0U^0 = -1}{U^0}, 0, 0, 0)$$

$$\Rightarrow \quad T_{00} \approx \rho \quad , \quad T^\alpha{}_\alpha \simeq -\rho$$

$$\boxed{R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \tfrac{1}{2} T^\alpha{}_\alpha\, g_{\mu\nu} \right)} \quad \xRightarrow[\mu=\nu=0]{} \quad \boxed{\nabla^2 \Phi \simeq 4\pi G \rho} \ ✓$$

Lagrangian formulation of GR

Hilbert action : $S_H = \int d^n x \sqrt{-g}\, R_{\mu\nu}\, g^{\mu\nu}$

$$\delta S_H = \delta S_1 + \delta S_2 + \delta S_3$$

$$\delta S_1 = \int d^n x \sqrt{-g}\; g^{\mu\nu}\; \delta R_{\mu\nu}$$

$$\delta S_2 = \int d^n x \sqrt{-g}\; R_{\mu\nu}\; \delta g^{\mu\nu}$$

$$\delta S_3 = \int d^n x \quad R\; \delta\sqrt{-g}$$

Firstly notice that:

$$g^{\mu\nu}\, g_{\nu\alpha} = \delta^{\mu}_{\alpha} \implies \delta g^{\mu\nu}\, g_{\nu\alpha} + g^{\mu\nu}\, \delta g_{\nu\alpha} = 0$$

$$\iff \boxed{\delta g^{\mu\nu} = -g^{\mu\alpha}\, g^{\nu\beta}\, \delta g_{\alpha\beta}}$$

Secondly, recall that

$$\begin{array}{ccccccc} \delta R^{\rho}{}_{\mu\lambda\nu} & = & \partial_\lambda \delta\Gamma^{\rho}{}_{\nu\mu} & - & \partial_\nu \delta\Gamma^{\rho}{}_{\lambda\mu} & + & \delta\Gamma\Gamma \\ & & \downarrow & & \downarrow & & \downarrow \text{ normal coord.} \\ \delta R^{\rho}{}_{\mu\lambda\nu} & = & \nabla_\lambda \underbrace{\delta\Gamma^{\rho}{}_{\nu\mu}}_{\text{tensor}} & - & \nabla_\nu \delta\Gamma^{\rho}{}_{\lambda\mu} & & 0 \end{array}$$

but this a tensorial equation !

$$\delta S_1 = \int d^n x \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} = \int d^n x \sqrt{-g}\, \underbrace{g^{\mu\nu} \left[\nabla_\lambda \delta\Gamma^{\lambda}{}_{\nu\mu} - \nabla_\nu \delta\Gamma^{\lambda}{}_{\lambda\mu} \right]}_{\nabla_\sigma \left[g^{\mu\nu} \delta\Gamma^{\sigma}{}_{\nu\mu} - g^{\mu\sigma} \delta\Gamma^{\lambda}{}_{\lambda\mu} \right]}$$

$$= \int d^n x \sqrt{-g}\, \nabla_\sigma J^\sigma = 0 \qquad (\text{up to boundary terms})$$

Finnaly,

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}\,\delta g \quad , \quad g = \det g_{\mu\nu}$$

Recall that

$$\log(\det M) = \mathrm{Tr}(\log M) \Rightarrow \frac{\delta \det M}{\det M} = \mathrm{Tr}(M^{-1}\delta M)$$

$$\Rightarrow \quad \delta g = g\, g^{\mu\nu}\,\delta g_{\mu\nu} = -g\, g_{\mu\nu}\,\delta g^{\mu\nu}$$

Putting everything together

$$\delta S_{H} = \int d^4x \sqrt{-g}\,\Big(\underbrace{R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}}_{\overset{!}{=}0}\Big)\,\delta g^{\mu\nu}$$

Einstein equation
in vacuum

In general, the full action is

$$S = \frac{1}{16\pi G} S_H + \overset{\text{matter}}{S_M}$$

Free discussion